EXACT EXPRESSIONS FOR A MULTIPLY DIFFRACTED WAVE WITH A CIRCULAR FRONT

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An exact expression in the form of a multiple integral is obtained for an acoustic wave which has a circular or straight front and which undergoes diffraction at the vertices of a polygon. Formulas are also obtained for any term of the geometric-acoustical expansion of this wave near its front.

1. Various representations of a wave having a front in the form of a circular arc. In the region $t \geqslant t_0 \geqslant 0,\ 0 < \rho < \infty,\ \theta_1 < \theta < \theta_2$ (ρ, θ are polar coordinates), we consider the solution $u(t, \rho, \theta)$ of the wave equation

$$u_{tt} = u_{\rho\rho} + \rho^{-1}u_\rho + \rho^{-2}u_{\theta\theta} \tag{1.1}$$

which is equal to zero for $\rho > t$ (i.e. ahead of the front) and which is homogeneous of degree zero in t and ρ. A wave from a point source of constant intensity which is cut in at $t = 0$ and the diffracted wave in problems of diffraction of a plane wave by a wedge [1 and 2] are examples of solutions of this type. According to [1], such a solution is representable, for $\rho < t$, in the form

$$u = \operatorname{Re} U(\zeta), \qquad \zeta = \left[\frac{t}{\rho} - \left(\frac{t^2}{\rho^2} - 1\right)^{1/2}\right] e^{i\theta} \tag{1.2}$$

where $U(\zeta)$ is an analytic function of the complex variable ζ which is purely imaginary on the arc $\zeta = e^{i\theta}, \theta_1 < \theta < \theta_2$. Conversely, for any such function $U(\zeta)$, Equation (1.2) provides a solution of Equation (1.1) which possesses the properties indicated. Setting

$$\zeta = e^{i(\theta + i\eta)}, \qquad U(\zeta) = U_1(\theta + i\eta) \tag{1.3}$$

in (1.2) and taking into consideration that $u = 0$ for $\rho = t$ (i.e. for $\eta = 0$), we obtain that $u = \operatorname{Re} U_1(\theta + i\eta)$ is an odd function of η and that

$$u = \frac{U_1(\theta + i\eta) - U_1(\theta - i\eta)}{2} = i\eta\, U_1'(\theta) - \frac{i\eta^3}{3!} U_1'''(\theta) + \ldots \tag{1.4}$$

Using Equations (1.2) and (1.3) and setting $\tau = t - \rho$, we obtain

$$\eta = \int_1^{t/\rho} \frac{dz}{\sqrt{z^2-1}} = \int_0^{\tau/\rho} \left[2s\left(1+\frac{s}{2}\right)\right]^{-1/2} ds = \left(\frac{2\tau}{\rho}\right)^{1/2} + \sum_{j=1}^{\infty} c_j \left(\frac{\tau}{\rho}\right)^{j+1/2}$$

The equations which have been written imply that

$$u = \sum_{j=0}^{\infty} \frac{a_j(\theta)}{\rho^{j+1/2}} f_j(\tau) \tag{1.5}$$

$$f_j(\tau) = \tau^{j+1/2}/\Gamma(j + 3/2), \qquad a_0(\theta) = i\sqrt{\pi/2}\, U_1'(\theta) = -\sqrt{\pi/2}\, e^{i\theta} U'(e^{i\theta}) \tag{1.6}$$

According to Section 8 of the paper [3] the series (1.5) must have the same form as the series (8.2) of [3]. Therefore, (the $L_{2j}{}^{\theta}a(\theta)$ are the same as in [3])

$$a_j(\theta) = \frac{(-1)^j}{2^j j!} L_{2j}{}^{\theta} a(\theta), \qquad a(\theta) = a_0(\theta) \tag{1.7}$$

The series (1.5) gives the geometric-acoustical expansion (ray expansion) of the wave of form (1.4).

It follows from (1.4) to (1.7) that for an analytic function $a(\theta + i\eta)$ which is real for $\eta = 0$, we have

$$\sum_{j=0}^{\infty} \frac{(-1)^j L_{2j}{}^{\theta} a(\theta)}{2^j j!\, \rho^{j+1/2}} \frac{\tau^{j-1/2}}{\Gamma(j+1/2)} = \frac{a(\theta + i\eta) + a(\theta - i\eta)}{\sqrt{2\pi}} \frac{\partial \eta}{\partial \tau} \tag{1.8}$$

It is easy to see that this equation is also valid for any analytic function $a(\theta + i\eta)$.

2. Investigation of the solution near the boundary. In the sector $\theta_1 < \theta < \theta_2$ let us consider the solution (1.2) of Equation (1.4) which satisfies the boundary condition

$$\partial u / \partial n = c\, \partial u / \partial t \qquad (c = \text{const}, \quad 0 \leqslant c \leqslant \infty) \tag{2.1}$$

for $\theta = \theta_1$.

Here $\partial u / \partial n = \rho^{-1} \partial u / \partial \theta$ in the inner normal derivative to the boundary $\theta = \theta_1$ of the sector. In particular, for $c = 0$ the condition (2.1) reduces to the condition $\partial u/\partial n = 0$ and for $c = \infty$ to the condition $\partial u/\partial t = 0$, i.e. to the condition $u = 0$, since for $t \leqslant \rho$ we have $u = 0$ for the solution (1.2). For a steady-state oscillation $u = ve^{i\omega t}$, condition (2.1) reduces to Leontovich's well-known condition $\partial v / \partial n = ic\omega v$.

For the solution (1.5), the boundary condition assumes the form

$$\operatorname{Re}\,(c + \sin i\eta)\, a(\theta_1 + i\eta) = 0 \quad \text{for } \eta \geqslant 0 \tag{2.2}$$

Therefore, the function $a(\theta + i\eta)$ is continued analytically into the region $2\theta_1 - \theta_2 < \theta < \theta_1$, in accordance with Equation

$$[c + \sin(\theta - \theta_1 + i\eta)]\, a(\theta + i\eta) \equiv \psi(\theta + i\eta)$$

$$\psi(\theta + i\eta) = -\operatorname{Re}\psi(2\theta_1 - \theta + i\eta) + i \operatorname{Im}\psi(2\theta_1 - \theta + i\eta)$$

It follows from this that $(c+\sin\varphi)\,a(\theta_1+\varphi)$ is an odd function of φ

$$(c+\sin\varphi)\,a(\theta_1+\varphi)\equiv-(c-\sin\varphi)\,a(\theta_1-\varphi),\qquad \varphi=\theta-\theta_1 \qquad (2.3)$$

Therefore, in the case of condition (2.1) for $\theta=\theta_1$ ($c\neq 0$) all derivatives $a^{(2n)}(\theta_1)$ are expressed in terms of $a', a''', \ldots a^{(2n-1)}$ (2.4)

$$a(\theta_1)=0,\quad a''(\theta_1)=-2c^{-1}a'(\theta_1),\quad a^{IV}(\theta_1)=4c^{-1}(a'(\theta_1)-a'''(\theta_1)),\ \ldots$$

3. Reflection from a boundary. In the region $y>0$ let the wave $u_1=\mathrm{Re}\,U(\zeta_1)$ propagate, the wave having a circular front with center at (x_1, y_1), $y_1>0$, and let the condition (2.1) be stipulated on the boundary $y=0$, where $\partial/\partial n=\partial/\partial y$. We shall seek the reflected wave by the method of Sobolev [1] in the form $v=\mathrm{Re}\,V(\zeta_2)$, where

$$\zeta_k=\left[\frac{t}{\rho_k}-\left(\frac{t^2}{\rho_k^2}-1\right)^{1/2}\right]e^{i\theta_k},\quad x-x_1=\rho_k\cos\theta_k,\quad(-1)^k y+y_1=\rho_k\sin\theta_k \quad (k=1,2)$$

Then $\zeta_1=\zeta_2$ for $y=0$. Setting $u=u_1+v$ in (2.1), we obtain

$$\mathrm{Re}\,[(V'-U')\,i\,(1-\zeta^2)+2c\zeta\,(V'+U')]=0 \quad \text{for } \mathrm{Im}\,\zeta>0 \qquad (3.1)$$

Therefore, the expression in square brackets can only be equal to Bt, where B is a real constant. Noting that for $\zeta=e^{i\theta}$, U and V are purely imaginary quantities (see Section 1), we find $B=0$ and

$$V'(\zeta)=\frac{i(1-\zeta^2)-2c\zeta}{i(1-\zeta^2)+2c\zeta}\,U'(\zeta) \qquad (3.2)$$

Taking any ζ_0 such that $|\zeta_0|=1$ (the arbitrariness in the choice of $\arg\zeta_0$ does not affect the quantity v), we may write

$$V(\zeta)=\int_{\zeta_0}^{\zeta}V'(z)\,dz,\qquad v=\mathrm{Re}\,V(\zeta_2) \qquad (3.3)$$

In particular, if u_1 is a wave emanating from a source of unit intensity which starts at the instant $t=0$ at the point (x_1, y_1), then

$$u_1=\frac{1}{2\pi}\ln\left[\frac{t}{\rho_1}+\left(\frac{t^2}{\rho_1^2}-1\right)^{1/2}\right],\qquad U(\zeta_1)=-\frac{1}{2\pi}\ln\zeta_1$$

and we obtain for the reflected wave $v=\mathrm{Re}\,V(\zeta_2)$, by setting $c=\cos\gamma$,

$$V(\zeta)=-\frac{1}{2\pi}\left(\ln\zeta+2i\cot\gamma\ln\frac{\zeta e^{i\gamma}+i}{\zeta+ie^{i\gamma}}\right) \qquad (3.4)$$

Let us now write out the ray expansion for the reflected wave. The ray expansion for the incident wave $u_1=\mathrm{Re}\,U(\zeta_1)$ has the form (1.5) to (1.7), where u, ζ, ρ and θ should be replaced by u_1, ζ_1, ρ_1 and θ_1. The reflected wave $v=\mathrm{Re}\,V(\zeta_2)$ is obtained from the incident wave by substituting V and ζ_2 for U and ζ_1. Therefore, the ray expansion for the reflected wave has the form

$$v=\sum_{j=0}^{\infty}\frac{(-1)^j L_{2j}b(\theta_2)}{2^j j!\,\rho_2^{j+1/2}}\,f_j(\tau),\qquad b(\theta_2)=-\sqrt{\frac{\pi}{2}}\,e^{i\theta_2}V'(e^{i\theta_2}) \qquad (3.5)$$

From (3.5), (3.2) and (1.6) we obtain

$$b(\theta) = k(\theta)\, a(\theta), \qquad k(\theta) = \frac{\sin\theta - c}{\sin\theta + c} \tag{3.6}$$

4. Diffraction of a wave with a circular front. Let the wave (1.2) with a citcular front and center at the point O_0 undergo diffraction by an angle with vertex O_1. In this Section it will be assumed that the point O_0 does not lie on a side of the angle. Boundary conditions of the type (2.1) are specified on the sides of the angle. The values of the coefficients c are not necessarily the same on the two sides of the angle. Let (ρ,θ) and (r,φ) be polar coordinates with poles O_0 and O_1 and parallel polar axes; for the point O_1, we have $r = R$, $\varphi = \beta$.

According to Section 2 of [3], the solution of this problem is the sum of the incident, reflected and refracted waves. A method of forming the reflected wave has been explained above. The diffracted wave was obtained in [3] in the form of a ray expansion, the series (8.4), which converges near the front. We shall express the sum of this series in the form of an integral. To this end, we write the series (8.4) of [3] in the following form, taking into account the form of the function f_j in (1.5), (1.6) and using the notation $a(\pi+\beta)\, m(\varphi,\beta) = q(\beta,\varphi)$

$$w = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{j+k} L_{2k}{}^{\beta} L_{2j}{}^{\varphi} q(\beta,\varphi)}{2^{j+k} j!\,k!\,R^{k+1/2} r^{j+1/2}} \frac{\tau^{j+k+1}}{\Gamma(j+k+2)} \tag{4.1}$$

It follows from (4.1) that

$$\frac{\partial w}{\partial t} = \int_0^{\tau} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{j+k} L_{2k}{}^{\beta} L_{2j}{}^{\varphi} q(\beta,\varphi)}{2^{j+k} j!\,k!\,R^{k+1/2} r^{j+1/2}} \frac{\chi^{j-1/2}(\tau-\chi)^{k-1/2}}{\Gamma(j+1/2)\,\Gamma(k+1/2)}\, d\chi \tag{4.2}$$

Applying Formula (1.8) to the series in (4.2) twice, we find

$$\frac{\partial w}{\partial t} = \frac{1}{2\pi}\int_0^{\tau} \frac{q^*(\beta + i\eta_0,\ \varphi + i\eta_1)\, d\chi}{\sqrt{(R+\tau-\chi)^2 - R^2}\,\sqrt{(r+\chi)^2 - r^2}} \tag{4.3}$$

$$\eta_0 = \eta\left(\frac{\tau-\chi}{R}\right), \qquad \eta_1 = \eta\left(\frac{\chi}{r}\right) = \ln\left[1 + \frac{\chi}{r} + \left(\left(1+\frac{\chi}{r}\right)^2 - 1\right)^{1/2}\right]$$

$$q^*(\beta + i\eta_0, \varphi + i\eta_1) = \Sigma\, q(\beta \pm i\eta_0, \varphi \pm i\eta_1)$$

(the sum is taken over all four combinations of the $\pm$ signs).

Either from Equation (4.3) or directly from (4.1) we can find that

$$w = \frac{1}{2\pi}\iint_{D_2} q(\beta + i\eta_0, \varphi + i\eta_1)\, d\eta_0 d\eta_1 \tag{4.4}$$

where D_2 is the region $R(\cosh\eta_0 - 1) + r(\cosh\eta_1 - 1) < \tau$.

We note that the function $w^* = \partial w/\partial t$ is the diffracted wave for the incident wave $u^* = \partial u/\partial t$, where u is a wave of the form (1.2). An example of this type is the wave u^* due to an instantaneously acting source (i.e. the source is cut in and then immediately afterward cut out)

$$u^* = \frac{1}{2\pi\sqrt{t^2-\rho^2}} \quad (t > \rho), \qquad u^* = 0 \quad (t < \rho) \tag{4.5}$$

Here

$$U(\zeta) = -(2\pi)^{-1} \ln \zeta, \qquad a(\theta) = (8\pi)^{-1/2}$$

In this way the right-hand side of (4.3) with $q(\beta, \varphi) = (8\pi)^{-1/2} m(\varphi, \beta)$ gives the solution w^* of the plane problem of diffraction of the wave (4.5) due to the source by a wedge under the boundary conditions (2.1).

The function $m(\varphi,\beta)$ is determined in Section 8 of [3]. A comparison of Equation (8.3) of [3] with Equations (1.4) to (1.7) of this paper results in

$$m(\varphi, \beta) = i\sqrt{\pi/2}U_0'(\varphi), \qquad \mathrm{Re}\, U_0(\varphi + i\eta) = w^\circ \tag{4.6}$$

where w° is the diffracted wave (for the same wedge) corresponding to the incident plane wave (3.1) of [3] .

In the case when Equation (1.1) is considered in the sector $\theta_1 < \varphi < \theta_2$, with the boundary condition $u = 0$ for $\varphi = \theta_1$ and $\varphi = \theta_2$, the function w° and U_0 can be found by the methods set forth in [1]. Then

$$m(\varphi, \beta) \equiv m(\beta, \varphi) = \frac{\sqrt{\pi/2}}{2(\theta_2 - \theta_1)} \sum_{k=1}^{4} (-1)^k \cot \frac{\pi(\varphi_1 - \gamma_k)}{2(\theta_2 - \theta_1)} \tag{4.7}$$

$$\gamma_1 = -\gamma_3 = \pi - \beta_1, \quad \gamma_2 = -\gamma_4 = \pi + \beta_1, \quad \varphi_1 = \varphi - \theta_1, \quad \beta_1 = \beta - \theta_1$$

In the case of the boundary condition $\partial u / \partial n = 0$, one need only change the signs of the first and fourth terms of the sum in (4.7).

For the case of the boundary condition (2.1), $m(\varphi,\beta)$ is determined from (4.6), where

$$U_0'(\varphi) = \frac{\nu \sin \nu\varphi}{\cos^2 \nu\varphi} \frac{dW}{d\zeta}\bigg|_{\zeta = \sec \nu\varphi} \qquad \left(\nu = \frac{\pi}{\psi}\right)$$

If the wedge at which diffraction takes place is situated the same way as in [2], then $dW/d\zeta$ is determined by Equation (11) of [2] for $\gamma = \pi - \beta$; here ψ and ζ are the same as in [2].

We note that we always have $m(\varphi, \beta) \equiv m(\beta, \varphi)$ (Expression (4.1) should not be altered if the positions of the source and the point of observation are interchanged).

5. Multiple diffraction. Let the wave u_0 having a front in the form of a circular arc with center O_0 be diffracted by an angle with vertex O_1 and straight sides on which boundary conditions of the form (2.1) are specified. The diffracted wave u_1 is one again diffracted by an angle with vertex O_2, and so on. The values of the coefficient c of (2.1) may be different on the various sides of the angles; the values $c = 0$ and $c = \infty$ will also be allowable. An expression is sought for the wave u_s which is obtained after diffraction at the vertices $O_1, \ldots, O_s$ (the same vertex may appear several times, with different subscripts, in the sequence $O_1, \ldots, O_s$).

This formulation includes the problems of diffraction by a slit, a segment, a polygon, or by several polygons arbitrarily situated in the plane (except in cases when a point of junction of wave fronts strikes a vertex). In the case of diffraction by a slit, representations of diffracted waves by multiple integrals were obtained in [4]. In the case of diffraction by a polygon, an approximate representation of the diffracted waves near the front only was obtained in [5].

Let r_k, φ_k be polar coordinates with pole O_k ($k = 0, 1, \ldots, s$) and parallel polar axes. Each point O_{k+1} has the coordinates R_k, $\beta_k + \pi$ in the systen with pole O_k . We denote by $u_s(\tau_s, r_s, \varphi_s)$ the wave with center O_s which is obtained as a result of diffraction of the wave u_0 by the angles $O_1, \ldots, O_s$; here $\tau_s \equiv t - R_0 - \ldots - R_{s-1} - r_s$. Let $\tau_0 = 0$ be the front of the incident wave u_0 and let $u_0 = 0$ ahead of the front. Then $\tau_s = 0$ is the front of the wave u_s, and for $\tau_s < 0$ we have $u_s = 0$. Let the incident wave u_0 be represented by Equations (1.5) to (1.7) for $\theta = \varphi_0$, $\rho = r_0$, $\tau = \tau_0$. Then the wave u_1 is represented by the ray expansion (8.4) of [3] near the front, where $\beta, \varphi, r, R, \tau$ are replaced by $\beta_0, \varphi_1, r_1, R_0, \tau_1$. By transforming this series to a form analogous to (4.1) we obtain a representation of the wave u_1 in the form of a sum of waves each of which is expressed by a series of the form (1.5), but with different $a_0(\theta)$ and $f_j(\tau)$. After diffraction at the vertex O_2, each of these waves again provides a wave of the form (4.1). By adding these latter we obtain a wave u_2. Considering diffraction of the wave u_2 at the vertex O_3, and then at the vertices $O_4, \ldots, O_s$, we obtain analogously (for $l = 0$)

$$u_s = \sum_{j_0=0}^{\infty} \ldots \sum_{j_s=0}^{\infty} b_{j_0\ldots j_s} f_{j_0+\ldots+j_s+s/2}(\tau_s) \tag{5.1}$$

$$b_{j_0\ldots j_s} = \frac{(-1)^{j_0+\ldots+j_s} L_{2j_0}^{\beta_0} \ldots L_{2j_{s-1}}^{\beta_{s-1}} L_{2j_s}^{\varphi_s} q_s}{2^{j_0+\ldots+j_s+l} j_0! \ldots j_s! \, R_0^{j_0+1/2} \ldots R_{s-1}^{j_{s-1}+1/2} r_s^{j_s+1/2}} \tag{5.2}$$

$$q_s = q_s(\beta_0, \ldots, \beta_{s-1}, \varphi_s) = \\ = a(\pi + \beta_0) m_1(\beta_0, \pi + \beta_1) \ldots m_{s-1}(\beta_{s-2}, \pi + \beta_{s-1}) m_s(\beta_{s-1}, \varphi_s)$$

Each function $m_i(\beta_{i-1}, \varphi_i)$ is determined analogously to $m(\beta, \varphi)$ in accordance with Equation (4.6), where w^0 is now the wave which arises upon diffraction of a plane wave (the same as in (4.6)) moving from O_{i-1} to O_i at the vertex O_i . The series (5.1) is absolutely and uniformly convergent in the vicinity of the front of the wave u_s . The width of the region of convergence decreases to zero as we approach points of junction of the wave fronts of u_s and u_{s-1} . Grouping the terms of the series, we obtain the ray expansion of the wave u_s

$$u_s = \sum_{n=0}^{\infty} A_{sn}(r_s, \varphi_s) f_{n+s/2}(\tau_s), \qquad A_{sn}(r_s, \varphi_s) = \sum_{j_0+\ldots+j_s=n} b_{j_0\ldots j_s} \tag{5.3}$$

By transforming the series (5.1) using the method of Section 4, we obtain

$$u_s = \frac{1}{2^l (2\pi)^{(s+1)/2}} \int_{D_{s+1}} \ldots \int q_s(\beta_0 + i\eta_0, \ldots, \beta_{s-1} + i\eta_{s-1}, \varphi_s + i\eta_s) \, d\eta_0 \ldots d\eta_s \tag{5.4}$$

The region of integration D_{s+1} is determined by the inequality

$$R_0(\cosh\eta_0 - 1) + \ldots + R_{s-1}(\cosh\eta_{s-1} - 1) + r_s(\cosh\eta_s - 1) < \tau_s \tag{5.5}$$

It is also possible to represent $\partial u_s/\partial t$ in the form of an s-fold integral, analogously to (4.3).

Equations (5.1) to (5.4) which have been derived, and also Equation (8.4) of [3], are valid for the case when none of the segments $O_0O_1, O_1O_2, \ldots, O_{s-1}O_s$ which constitute the path of a ray up to the vertex O_s lies on the boundary.

In the case when l of them lie on the boundary, the factor 2^{-l} appears in these formulas. This happens of the following reason. The wave u_2, for example, is caused by diffraction at the vertex O_2 of the waves u_1 and v_1 simultaneously, the latter being obtained by reflection of u_1 from the side of the angle O_2 visible from the point O_1. If the segment O_1O_2 lies on the boundary, then, as can be shown, $v_1 = u_1$ and only one-half of the wave obtained by diffraction at point O_1 should be considered as the incident wave for the vertex O_2.

Equation (5.4), as also (4.3) and (4.4), is valid not only in the vicinity of the front, but also in the entire region occupied by the diffracted wave u_s, except for these values of φ_s for which the function $m_s(\beta_{s-1}, \varphi_s)$ has a singularity, i.e. except for the radii which connect the point O_s with points of juncture of the wave front of u_s and other fronts.

To prove this we note that near the front the function (5.4) coincides with (5.1). Therefore, it satisfies Equation (1.1) and the boundary condition (2.1) near the front. But, since it is analytic, (5.4) satisfies these everywhere except for the radii indicated above. It remains to prove by induction that the wave u_s is precisely the wave which comes about as a result of diffraction of the wave u_{s-1} at the angle O_s. Let the wave u_{s-1} formed by diffraction at the point O_{s-1} be expressed by the formula obtained from (5.4) by substituting $s-1$ for s, and by $u_{s-1} = 0$ in the shadow zone (a shadow can be caused by the presence of an obstacle, the angle with vertex O_s). Using the fact that for $\varphi_s = \pi + \beta_{s-1}$, i.e. on the boundary of the shadow, the function $m(\beta_{s-1}, \varphi_s)$ has a pole with an easily computed residue, it can be shown that the sum $u_{s-1} + u_s$, where u_s is determined by (5.4), is continuous along with its first derivatives for $\varphi_s = \pi + \beta_{s-1}$. It follows from this that $u_{s-1} + u_s$ is a solution of Equation (1.1) in a region containing the radius $\varphi_s = \pi + \beta_{s-1}$.

The sum $v_{s-1} + u_s$ may be investigated similarly on the radius drawn from the point of juncture of the wave u_s and the front of the reflected wave v_{s-1}, if the latter exists. Here the following expression is used for the wave v_{s-1} obtained by reflection of the wave u_{s-1} from one side of the angle O_s

$$2^l (2\pi)^{s/2}\, v_{s-1}(\tau_{s-1}, r^*_{s-1}, \varphi^*_{s-1}) = \qquad (\varphi^*_{s-1} = 2\alpha - \varphi_{s-1}) \tag{5.6}$$

$$= \int \cdot \underset{D_s}{\cdot} \int q_{s-1}(\beta_0 + i\eta_0, \ldots, \varphi_{s-1} + i\eta_{s-1})\, k(\varphi_{s-1} - \alpha + h\pi + i\eta_{s-1})\, d\eta_0 \ldots d\eta_{s-1}$$

where α is the polar angle giving the direction of the straight line on which the reflection takes place; $r^*_{s-1}, \varphi^*_{s-1}$ are polar coordinates with pole at the point O^*_{s-1}, which is the point symmetrical to O_{s-1} with respect to this line; k is the reflection coefficient of (3.6); and h is the integer so that $0 < \varphi_{s-1} - \alpha + h\pi < \pi$. To prove Equation (5.6) it is sufficient to verify that the sum $u_{s-1} + v_{s-1}$ satisfies the boundary condition (2.1).

If the polygon on which diffraction occurs is not convex, there may exist waves which experience reflection from the sides of the polygon after a certain number of diffractions at its vertices and are then once again diffracted at vertices etc. Each such wave is again expressed by (5.1) and (5.4), but now

$$q_s(\beta_0, \ldots, \beta_{s-1}, \varphi_s) = a(\pi + \beta_0)\, m_1(\beta_0, \pi + \beta_1) \ldots m_j(\beta_{j-1}, \pi + \beta_j) \times$$
$$\times k(\beta_j - \alpha_j + h_j\pi)\, m_{j+1}(2\alpha_j - \beta_j, \pi + \beta_{j+1}) \ldots m_s(\beta_{s-1}; \varphi_s) \tag{5.7}$$

Thus, if a ray undergoes a reflection from a straight line which is characterized by the polar angle α_j on the path between vertices O_j and O_{j+1}, then the expression for q_s is modified by the factor $k(\beta_j - \alpha_j + h_j\pi)$, where $k(\theta)$ is the reflection coefficient of (3.6); h_j is the integer such that $0 < \beta_j - \alpha_j + h_j\pi < \pi$; and in the factor which follows (in the present case m_{j+1}), the first argument β_j is replaced by $2\alpha_j - \beta_j$, i.e. by a polar angle of the direction from which the reflected ray approaches. In the presence of several reflections, a corresponding number of factors k is appended. The proof is carried out analogously to the proof of Equations (5.4) and (5.6).

6. Investigation of the diffracted wave. The coefficients $A_{sn}(r_s, \varphi_s)$ of the ray expansion (5.3) have a form

$$A_{s0}=\frac{q_s}{2^l P_s}, \quad A_{s1}=-\frac{1}{2^{l+1}P_s}\sum_{j=0}^{s}\frac{L_2^{\beta j}q_s}{R_j}$$

$$P_s=(R_0\ldots R_{s-1}r_s)^{1/2}$$

$$A_{s2}=\frac{1}{2^{l+3}P_s}\Big(\sum_{j=0}^{s}\frac{L_4^{\beta j}q_s}{R_j^2}+2\sum_{0\leqslant j<k\leqslant s}\frac{L_2^{\beta j}L_2^{\beta k}q_s}{R_jR_k}\Big)$$

where l and q_s are the same as in (5.2), and β_s and R_s replace φ_s and r_s. For large n, Equations (5.3) and (5.2) for A_{sn} are unwieldy. It is sometimes more convenient to obtain several of the leading nonzero terms of the series (5.3) by finding the expansions of the form (5.3) or (6.2) (see below) successively for the waves $u_1,\ldots,u_s$.

Let the ray expansion (or several of its leading terms) be known for the wave u_{s-1} $(s\geqslant 1)$

$$u_{s-1}(\tau_{s-1},r_{s-1},\varphi_{s-1})=\sum_{n=0}^{\infty}A_{s-1,n}(r_{s-1},\varphi_{s-1})f_{d+n}(\tau_{s-1}) \tag{6.1}$$

According to Section 8 of [3], we may write

$$u_{s-1}=\sum_{n=0}^{\infty}F\,(r_{s-1},f_{d+n}(\tau_{s-1}),b_n(\varphi_{s-1})) \tag{6.2}$$

Here each term has the form

$$F\,(\rho,f_m(\tau),b(\theta))\equiv\sum_{j=0}^{\infty}\frac{(-1)^jL_{2j}^{\theta}b(\theta)}{2^jj!\,\rho^{j+1/2}}f_{m+j}(\tau) \tag{6.3}$$

and is, therefore, a solution of the wave equations; the functions $b_n(\varphi_{s-1})$ are different in the various terms. The convergence of the series (6.2) in the vicinity of the front is guaranteed by the assumption that

$$|\partial^m b_n/\partial\varphi_{s-1}^m|\leqslant Cm!n!\delta^{-m-n},\qquad \delta>0 \tag{6.4}$$

Diffraction of a wave of the form (6.3) at the vertex O_s results in a wave of the form (4.1), and diffraction of the wave u_{s-1} of the form (6.2) in the wave

$$u_s=\sum_{n=0}^{\infty}F\left(r_s,f_{d+n+1/2}(\tau_s),\quad\sum_{k=0}^{n}\frac{(-1)^kL_{2k}^{\beta_{s-1}}b_{n-k}(\pi+\beta_{s-1})\,m_s(\beta_{s-1},\varphi_s)}{2^{k+l}k!\,R_{s-1}^{k+1/2}}\right) \tag{6.5}$$

where $l=0$ if the segment $O_{s-1}O_s$ does not lie on the boundary and $l=1$ if it does; m_s is the same as in Section 5. The series (6.5) is convergent under the condition (6.4) near the wave front and an inequality similar to (6.4) is valid for it also. Thus, if the incident wave u_0 is written in the form (6.2) for $s=1$, and if the estimate (6.4) holds $b_n(\varphi_0)$, then $u_1,\ldots,u_s$ can be found successively in accordance with (6.5). The transformation from the expression of u_s in the form (6.5) to a ray expansion may be effected with the aid of Equation (6.3).

If the segment $O_{s-1}O_s$ lies on the boundary and the boundary condition there is specified to be $u=0$ or the condition (2.1) for $c\neq 0$, the first term of the sum in (6.5) vanishes. Let us find the subsequent terms. The boundary condition of $O_{s-1}O_s$ can be written as

$$\frac{\partial u}{\partial t} = -\frac{\alpha}{r_{s-1}}\frac{\partial u}{\partial \varphi_{s-1}} \quad (\varphi_{s-1} = \pi + \beta_{s-1}), \qquad \frac{\partial u}{\partial t} = \frac{\alpha}{r_s}\frac{\partial u}{\partial \varphi_s} \quad (\varphi_s = \beta_{s-1})$$

Here $\alpha = 1/c$ or $\alpha = -1/c$; in the case of the boundary condition $u = 0$ we have $\alpha = 0$. According to (2.4) we have on $O_{s-1}O_s$ the following relations for the functions $b = b_{n-k}(\pi + \beta_{s-1})$ and $m = m_s(\beta_{s-1}, \varphi_s)$:

$$b = 0,\ b'' = 2\alpha b',\ b^{\mathrm{IV}} = 4\alpha(b''' - b'), \ldots,$$

$$m = 0,\ m'' = -2\alpha m',\ m^{\mathrm{IV}} = 4\alpha(m' - m'''), \ldots . \tag{6.6}$$

where all derivatives are taken with respect to β_{s-1}. Therefore,

$$L_0 bm = 0,\ L_2 bm = 2b'm',\ L_4 bm = 4b'''m' + 4b'm''' + (5 - 24\alpha^2)\, b'm', \ldots$$

in Equation (6.5) and the ray expansion of the diffracted wave begins as follows:

$$u_s = -\frac{b_0'(\pi + \beta_{s-1})\, m_s'(\beta_{s-1}, \varphi_s)}{2R_{s-1}^{3/2} r_s^{1/2}} f_{d+3/2}(\tau_s) + \ldots$$

Thus, if the leading term of the ray expansion of the incident wave u_0 contains $f_0(\tau)$, the ray expansion for the diffracted wave contains $f_{p+1/2s}(\tau)$, where s is the number of vertices $O_1, \ldots, O_s$ encountered on the path of the ray and p is the number of segments $O_{k-1}O_k$ of the path of the ray which lie on a boundary having the boundary condition $u = 0$ or (2.1) for $c \neq 0$. This means that the smoothness of onset of the wave u_s is greater by $p + \frac{1}{2}s$ units than for the incident wave u_0 ; i.e. if $u_0 \sim b_0\tau^m$ near the front, then $u_s \sim b_s\tau^{m+p+1/2s}$.

7. Steady-state diffraction. For a steady-state oscillation of the form $u(t, x, y) = v(x, y)\, e^{i\omega t}$ one usually obtains the high-frequency asymptotic expression $(\omega \to \infty)$ by setting $f_m(\tau) = (i\omega)^{-m} e^{i\omega\tau}$ in the ray expansion for u. In the case of diffraction being considered here, the leading term of the asymptotic expansion of the wave u_s (for $\omega \to \infty$) acquires an amplitude factor of $\omega^{-p-1/2s}$ and a phase lag of $(p + \frac{1}{2}s)\pi/2$ relative to u_0, where p is the number of segments of the path of the ray which lie on a boundary having either the boundary condition $v = 0$ or the impedence boundary condition $\partial v / \partial n = ic\omega v,\ c \neq 0$.

Thus, for the problems of diffraction by polygons, formulated at the beginning of Section 5, the method which has been explained allows us to write out in a finite number of operations the exact expressions for the coefficients of an asymptotic expansion of the solution $v(x,y)$ in powers of $1/\omega$ up to $1/\omega^n$, for any fixed n . To do this, it is sufficient to examine all optical paths $O_0O_1, \ldots, O_s$ for which the ray expansion of t' diffracted wave starts with a term $f_m(\tau)$, $m = p + 1/2\, s \leqslant n$. There are obviously, a finite number of such paths.

In case of a steady-state oscillation the diffracted wave can also be represented in the form of an integral analogous to (5.4). If $u(t, \rho, \theta)$ is a solution of Equation (1.1) $u_{tt} = \triangle u$, then the function

$$V(\rho, \theta) = \int_{-\infty}^{\infty} e^{-i\omega t}\, u_t(t, \rho, \theta)\, dt \tag{7.1}$$

satisfies the equation $\triangle V + \omega^2 V = 0$ (it is assumed that the derivatives of u fall off sufficiently rapidly for $t \to \pm\infty$ so that it is possible to differentiate under the integral sign and to integrate by parts). Equations (7.1) and (1.8) give us

$$V_0(\rho, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\rho \cosh \eta} a(\theta + i\eta)\, d\eta \tag{7.2}$$

for the wave (1.5) to (1.7).

If $a \equiv (8\pi)^{-1/2}$, u_t coincides with the wave (4.5) coming from an instantaneously acting source and $e^{i\omega t}V_0 = -{}^1/_4 i e^{i\omega t} H_0^{(2)}(\omega r)$ is a wave from a point source of intensity $e^{i\omega t}$; , where $H_0^{(2)}$ is a Hankel function.

If, however, $a(\theta) \equiv m(\beta, \theta)$, where the function m is the same as in (4.7), then $e^{i\omega t}V_0$ is the diffracted wave corresonding to the plane wave $e^{i\omega t} e^{i\omega(x\cos\beta + y\sin\beta)}$ incident on a wedge. The change of variable integration $\theta + i\eta = \psi$ reduces (7.2) to Sommerfeld's integral (the difference in paths of integration is related to that fact that Sommerfeld's integral gives the entire field consisting of the diffracted wave (7.2), the incident and reflected waves).

For the multiply diffracted wave (5.4), Equation (7.1) gives us

$$V_s(r_s, \varphi_s) = 2^{-l} (2\pi)^{-(s+1)/2} \int \ldots \int e^{-i\omega z} q_s \, d\eta_0 \ldots d\eta_s \tag{7.3}$$

$$z = R_0 \cosh \eta_0 + \ldots + R_{s-1} \cosh \eta_{s-1} + r_s \cosh \eta_s$$

where q_s is the same as in (5.4) and the integration on $\eta_0, \ldots, \eta_s$ is carried out from $-\infty$ to $+\infty$. It can be proved that the integral converges and admits two differentiations. Therefore, V_s is the wave obtained by diffraction of the wave (7.2) at the vertices $O_1, \ldots, O_s$.

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